A crossword-problem equivalent to the continuum hypothesis

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Let A be any set and D be a set of functions $A \to \{0,1\}$. A D-crossword is a function $A \times A \to \{0,1\}$ having all of its rows and columns in D. We say that D is crossword-closed if the diagonal of every D-crossword is in D.

In this note we concentrate on the crossword-closed sets in the continuum having Tychonoff-type separation properties and show that there is a statement about these that is equivalent to the continuum hypothesis. But let us first review the countable case.

1 Countable crosswords

Let $A = \omega$. We say that D has the finite-T_1 property if for any finite set of points $x_n \in A$ and values $y_n \in \{0,1\}$ there is a function $u \in D$ such that $u(x_n) = y_n$.

Theorem 1. *If D is crossword-closed and has the finite-T_1 property, then* $D = 2^A$.

Proof. (Idea of this proof is from Mark Aujus.) Let D be crossword-closed and finite-T_1. For any $w : \omega \to \{0,1\}$ construct a symmetric relation $r \subset \omega_1 \times \omega_1$ (we don't distinguish between bit sequences and sets) by $r = \bigcup_{n<\omega} F(\alpha)$ where $F : \omega \to 2^{\omega\times\omega}$, $F(0)$ is the empty relation added some row $u \subset \omega$ such that $u(0) = w(0)$ at place 0

$$\begin{aligned} F(n+1) = F(n) \cup \{(n+1, m) \mid\ & m < n+1 \implies (m, n+1) \in F(\alpha), \\ & m = n+1 \implies m \in w, \\ & m > n+1 \implies m \in \{\text{a set such that this row is in D}\}\}, \end{aligned}$$

Now, using the fact that every ordinal smaller than ω is finite, and the finite-T_1 property of D, we see that r has w in the diagonal and every row and column in D. $\square$

2 Crosswords in the continuum

Let $A = [-1, 1] \subset \mathbb{R}$. We say that D has the countable-T_1 property if for any sequence of points $x_n \in A$ and values $y_n \in \{0, 1\}$ there is a function $u \in D$ such that $u(x_n) = y_n$.

Theorem 2. *The continuum hypothesis is equivalent to the following: if* D *is crossword-closed and has the countable-*T_1 *property, then* $D = 2^A$.

Proof. First, assume the continuum hypothesis. Let D be crossword-closed and countable-T_1. First, use the axiom of choice and the continuum hypothesis to obtain a bijection from A to the first uncountable ordinal ω_1. Now if w is any function $\omega_1 \to \{0, 1\}$, apply transfinite recursion up to ω_1 to construct a symmetric D-crossword having w in the diagonal.

To be more precise, let us construct the symmetric relation $r \subset \omega_1 \times \omega_1$ by $r = \bigcup_{\alpha<\omega_1} F(\alpha)$ where $F : \omega_1 \to 2^{\omega_1 \times \omega_1}$, $F(0)$ is the empty relation added some row $u \subset \omega_1$ such that $u(0) = w(0)$ at place 0, for a successor ordinal

$$\begin{aligned} F(\alpha+1) = F(\alpha) \cup \{(\alpha+1, \beta) \mid \beta < \alpha+1 &\implies (\beta, \alpha+1) \in F(\alpha), \\ \beta = \alpha+1 &\implies \beta \in w, \\ \beta > \alpha+1 &\implies \beta \in \{\text{a set such that this row is in D}\}\}, \end{aligned}$$

and for a limit-ordinal $F(\alpha) = \bigcup_{\beta<\alpha} F(\beta)$. Now, using the fact that every ordinal smaller than ω_1 is countable, and the countable-T_1 property of D, conclude that r has w in the diagonal and every row and column in D.

Next, assume the negation of the continuum hypothesis. Then define D to be the set of all functions $A \to \{0, 1\}$ that differ from the unit step function

$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geqslant 0 \end{cases}$$

only at a countable number of points. Now D clearly has the countable-T_1 property. It is also vacuously crossword-closed since for any D-crossword r we would have: (this goes along with Freiling's [1] ideas for the most part) Let $s : [-1, 0] \times [0, 1] \to \{0, 1\}$ be the confinement of r to the set $[-1, 0] \times [0, 1]$ (meaning that $s(x, y) = r(x, y)$ when $(x, y) \in [-1, 0] \times [0, 1]$). Now the horizontal sections $s_x = \{y \mid s(x, y) = 1\}$ are countable, and the complements of the vertical sections $s^y = \{x \mid s(x, y) = 1\}$ in $[0, 1]$ are countable. Define $f(x) = s_{x-1} \cup ([0, 1] \setminus s^x)$ and notice that for every $x \in [0, 1]$, the set $f(x)$ is countable. By [1] the negation of the continuum hypothesis implies the Freiling's axiom of symmetry: for every $f : [0, 1] \to 2^{[0,1]}$ such that $f(x)$ is countable for all $x \in [0, 1]$ there exists $x, y \in [0, 1]$ such that $x \notin f(y)$ and $y \notin f(x)$. Now let x and y be such points for f. Then by definition of f we have $x \notin s_{y-1}$, $x \in s^y$, $y \notin s_{x-1}$, $y \in s^x$, and by definition of sections we have $x \in s_{y-1} \iff y \in s^x$, which is a contradiction.

Finally, it is clear that $D \neq 2^A$. □

Acknowledgements

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References

[1] Chris Freiling: *Axioms of Symmetry: Throwing Darts at the Real Number Line*, J. Symbolic Logic, Volume 51, Issue 1 (1986), 190-200.